The Renormalization Group and Fractional Brownian Motion

David Hochberg[*] and Juan Pérez–Mercader[†]

Centro de Astrobiología, (CSIC-INTA),

28850 Torrejón de Ardoz, Madrid, Spain.

(Dated: February 1, 2008)

Abstract

We find that in generic field theories the combined effect of fluctuations and interactions leads to a probability distribution function which describes fractional Brownian Motion (fBM) and "complex behavior". To show this we use the Renormalization Group as a tool to improve perturbative calculations, and check that beyond the classical regime of the field theory (i. e., when no fluctuations are present) the non–linearities drive the probability distribution function of the system away from classical Brownian Motion and into a regime which to the lowest order is that of fBM. Our results can be applied to systems away from equilibrium and to dynamical critical phenomena. We illustrate our results with two selected examples: a particle in a heat bath, and the KPZ equation.

---

[*]Electronic address: hochberg@laeff.esa.es

[†]Electronic address: mercader@laeff.esa.es

Complex behavior is ubiquitous [1]. From fluids to ecosystems to chemistry, we are familiar with phenomena often associated with non–gaussian Probability Distribution Functions (PDFs); phenomena described by PDFs with "long tails" or "stretched exponential" behaviors [2]. The PDFs associated with these phenomena have the property that, typically, [3] "improbable (very bad) events are much more likely than with a Gaussian". Many of these phenomena are more or less loosely associated with, e. g., "complexity phase transitions" [4] and "fractal behavior" [5]; furthermore, often, "what is seen depends on the size of the observer" [3] and is accompanied by the property that "one law leads to many behaviors" [3]. They occur in systems that are extended in space and in time, with many interacting components and where both, "the random and the regular" [6], are at work.

Here we show how this "complex behavior" can be understood as a natural consequence of space–time evolution, fluctuations and interactions in a many body system observed at different scales and/or with varying initial conditions on the parameters describing the system. We do this for *a generic* many body system: a (quantum, thermal or stochastic) field theory where space–time evolution and interactions represent the "regular" and fluctuations represent the "random".

The logic we use is the following: as is well known [7], in any field theory fluctuations and interactions (a) modify the n-point correlation functions of the field at different space-time points, and (b), lead to divergences in the original set of parameters such as couplings and diffusion constants, masses, etc. defining the system. From (a) it follows that the *characteristic function* of the field theory is modified from its originally normal and gaussian character, and therefore the PDF itself is also modified; furthermore, (b) leads to a scale-dependence of the parameters and correlation functions which, asymptotically, are power laws [8] with exponents calculable within the framework of the Renormalization Group (RG) [9], [10].

Specifically, we address here the problem of how interactions and fluctuations manifest themselves at *asymptotic scales* on the system's probability distribution and discover that (i) the specific asymptotic properties of the PDF depend on the basins of attraction of various RG fixed points and (ii) that, asymptotically, the field theory displays the type of behavior we have called "complex" above.

We begin by introducing the random field $\phi(\vec{x}, t_x)$. We coarse-grain (or filter) this field by means of a "window-function" W_R which averages out the small scale features in ϕ; here

we mean small in comparison to a window length scale R. Without loss of generality, we take the window function to be translationally invariant and define

$$\phi_R(y) = \frac{1}{V} \int dx \, W_R(y-x) \, \phi(x), \tag{1}$$

where it is understood that $x = (\vec{x}, t_x)$ where $\vec{x}$ is a d-dimensional vector, and $V = \Omega T$ is the spatial volume Ω times the time interval T over which the field is filtered. Typical window functions are the "top-hat" window $W_R(x) = \Theta(R - |x|)$ and the Gaussian window $W_R(x) = \exp(-|x|^2/2R^2)$. In (1) the coarse-grained field $\phi_R(y)$ is the result of averaging ϕ over a space-time region or "cell" of linear extent R, "centered" at space-time point y, over the time interval T. The small scale features of the field are blurred within the space-time region selected by the window: the resolution is degraded. We also note that (1) is a linear transformation relating the microscopic (ϕ) to the coarse-grained field (ϕ_R) that need not be invertible.

We can calculate [11, 12] the probability that the coarse-grained field $\phi_R(y)$ takes the *value* φ within the window:

$$p(\varphi; R, y) = \langle \delta(\phi_R(y) - \varphi) \rangle_P \equiv \int [\mathcal{D}\phi] \, \delta(\phi_R(y) - \varphi) \, P[\phi] \,, \tag{2}$$

where $P[\phi]$ is the PDF for the microscopic field configurations and the integral is a path integral over all field configurations. We point out that $\int d\varphi \, p(\varphi) = \int [\mathcal{D}\phi] \, P[\phi] = 1$, so that p is normalized to unity if and only if the PDF is. We see moreover that p is a probability density and scales dimensionally as the inverse field φ^{-1}. We will make use of this fact below. The above probability p can be easily related to $Z[J]$, the characteristic functional with source J (more commonly known as the generating functional) for the n–point functions. In fact, by using the Fourier integral representation of the delta distribution in Eq. (2) we have

$$\begin{aligned}
p(\varphi; R, y) &= \int d\xi \int [\mathcal{D}\phi] \, e^{i\xi[\phi_R(y) - \varphi]} P[\phi] \\
&= \int d\xi \, e^{-i\xi\varphi} \int [\mathcal{D}\phi] \, e^{i\frac{\xi}{V} \int dx W_R(x-y)\phi(x)} P[\phi] \\
&= \int_{-\infty}^{\infty} d\xi \, e^{-i\xi\varphi} Z[J_R(x-y) = i\frac{\xi}{V} W_R(x-y)] \,,
\end{aligned} \tag{3}$$

where Z in the last line is the *function* of ξ that results from evaluating the generating functional for a window-source function. For a *generic* gaussian probability functional [7], [13] and [14]

$$P[\phi] = \mathcal{N} \exp \left\{ -\frac{1}{2} \int dx \int dz \, \phi(x) G^{-1}(x,z) \phi(z) \right\} \ , \tag{4}$$

the path integral in Eq.(3) is gaussian ($\mathcal{N}$ is a normalization factor) and one immediately obtains

$$p(\varphi; R, y) = \sqrt{\frac{V^2}{2\pi[W_R \cdot G \cdot W_R]}} \cdot \exp \left(-\frac{\varphi^2}{\frac{2}{V^2}[W_R \cdot G \cdot W_R]} \right) \ . \tag{5}$$

where the compact notation $[W_R \cdot G \cdot W_R] = \int dx \int dz \, W_R(x-y) G(x,z) W_R(z-y)$ stands for the *filtered* or coarse-grained two-point correlation function associated with the microscopic field. Note that $p(\varphi)$ in (5) is normalized to unity. We will calculate the asymptotic scaling form of $[W_R \cdot G \cdot W_R]$ below. Here $G(x,z)$ is the "variance" (or "dispersion") of the microscopic field between space–time points x and z. More generally [13], $G(x,z)$ is the connected 2–point correlation function (or propagator), for the field ϕ, $\langle \phi(x)\phi(z) \rangle \equiv G(x,z)$.

In the presence of interactions and fluctuations the characteristic functional $Z[J]$ is modified. It can be obtained as an expansion in terms of the n–point correlation functions for the field $\phi(\vec{x}, t)$, which themselves are corrected due to interactions and fluctuations. In fact, as is well known [15], the n–point functions obey Renormalization Group equations (which follow [7], [10] from the fact that removal of divergences in the model introduces an arbitrary scale μ) describing how n–point functions change as the parameters– the couplings $\{g_j\}$ and mass m– in the model, or *the scale at which the system is observed*, are modified. The solution to the RG equations are the so-called "improved" n-point functions. For the connected n–point correlation function in momentum space, the RG equation in a mass-independent subtraction procedure is

$$\left[\mu \frac{\partial}{\partial \mu} + \sum_i \beta_i \frac{\partial}{\partial g_i} + \delta(g) m \frac{\partial}{\partial m} + \frac{1}{2} n \gamma_\phi \right] G^{(n)}_{Imp}(q; ...) = 0 \ . \tag{6}$$

Here, γ_ϕ is the anomalous dimension of $\phi(x,t)$, given by $\gamma_\phi = \mu(\partial \ln Z_\phi / \partial \mu)$ where Z_ϕ is the wave function renormalization constant of ϕ, $\delta(g) = -\mu(\partial \ln Z_m / \partial \mu)$ where Z_m is the mass renormalization constant, and q represents the momentum and frequency variables.

4

A RG equation for the improved probability density can be derived directly from Z as follows. Assuming a renormalizable field theory, the relation between the bare and renormalized generating functional is $Z[J_0, \{g_{0i}\}, m_0, \Lambda] = Z[J, \{g_i\}, m, \mu]$. The functional written in terms of the bare parameters and bare source (all having the zero-subscript) and cut-off Λ does not know about the arbitrary finite scale parameter μ , so that $\mu \frac{dZ[J,\{g_i\},m,\mu]}{d\mu} = 0$. The chain-rule immediately implies that

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_j \beta_j \frac{\partial}{\partial g_j} + \delta(g_i) m \frac{\partial}{\partial m} + \frac{\gamma_\phi}{2} \int dx \, J(x) \frac{\delta}{\delta J(x)} \right) Z[J, \{g_i\}, m, \mu] = 0, \qquad (7)$$

where the coefficient functions

$$\mu \frac{\partial g_j}{\partial \mu} = \beta_j(g), \qquad (8)$$

$$\mu \frac{\partial m}{\partial \mu} = \delta(g_i) \, m, \qquad (9)$$

$$\mu \frac{\partial J(x)}{\partial \mu} = \frac{\gamma_\phi}{2} \, J(x), \qquad (10)$$

describe the scale dependence for the couplings g_j, the mass m (if there is any) and the source function J. We emphasize that the renormalization of the source is equivalent to wavefunction renormalization, an important fact that is emphasized by Brown [16]. The source function acts as just another bare "parameter" of the theory, and it gets renormalized along with the other couplings. This RG equation for $Z[J]$ holds for arbitrary source functions. Due to (3) we now substitute $J(x) \rightarrow \frac{i\xi}{V} W_R(x)$, into (7) and then by means of the identity (proof: expand Z[J] out in a functional Taylor series in powers of J),

$$\int dx \, J(x) \frac{\delta}{\delta J(x)} Z[J] = \int dx \, W_R(x) \frac{\delta}{\delta W_R(x)} Z[i\xi W_R/V] = \xi \frac{\partial}{\partial \xi} Z[i\xi W_R/V], \qquad (11)$$

we obtain

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_j \beta_j \frac{\partial}{\partial g_j} + \delta(g_i) m \frac{\partial}{\partial m} + \frac{\gamma_\phi}{2} \xi \frac{\partial}{\partial \xi} \right) Z[i\xi W_R/V, \{g_i\}, m, \mu] = 0. \qquad (12)$$

Following (3) we Fourier transform this to arrive at the RG equation for p_{Imp}:

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_j \beta_j \frac{\partial}{\partial g_j} + \delta(g_i) m \frac{\partial}{\partial m} - \frac{\gamma_\phi}{2} \frac{\partial}{\partial \varphi} \varphi \right) p_{Imp}(\varphi; \{g_i\}, m, \mu; R) = 0, \qquad (13)$$

which follows after an integration by parts to eliminate ξ-derivatives in favor of φ-derivatives. Just as does equation (6), this equation expresses the independence of the *physics* on the choice of scale $\mu \sim \frac{1}{R}$ at which we defined the values of the coupling constants g_i of the

5

model. The coefficient functions β_j, which are calculable using perturbation theory, describe the scale dependence for each of the couplings according to (8). The scale μ is known as the "sliding" scale, and represents the scale at which the system is sampled. A hurricane looks very different depending on whether it is seen by a fly trapped inside it or by an astronaut from outer space; in one case $\mu_{Fly} \sim 1/\ell_{Fly}$, where ℓ_{Fly} is the typical length scale for a fly and correspondingly for the astronaut, where now $\ell_{Astronaut}$ is the size of the region of the Earth observed by the astronaut. $\varphi(\mu)$ satisfies the differential equation $\mu \frac{d\varphi(\mu)}{d\mu} = -\frac{1}{2}\gamma_\phi \varphi(\mu)$ (compare this to (10)) and represents the value of the coarse-grained field at scale μ. The explicit relation between μ and R is discussed in more detail below.

The solutions to equations (6) (for $n = 2$) and (13) are respectively given by

$$G_{Imp}(q; ...; g_j(\mu), m(\mu); \mu) =$$

$$e^{-\int_{\mu_0}^{\mu} \gamma_\phi(u)\, d\ln u} \cdot G_{Imp}(q_0; ...; g_j(\mu_0), m(\mu_0); \mu_0) \,, \tag{14}$$

and

$$p_{Imp}(\varphi(\mu), \{g_i(\mu)\}, m(\mu), \mu; R) = e^{\frac{1}{2}\int_{\mu_0}^{\mu} \gamma_\phi(u)\, d\ln u}\, p_{Imp}(\varphi(\mu_0), \{g_i(\mu_0)\}, m(\mu_0), \mu_0; R). \tag{15}$$

These solutions are to be interpreted as follows: for example, to obtain the improved [17] 2–point correlation function G_{Imp}, one needs to write down its explicit form at some scale μ_0 where it is known, and the values of the various couplings g_i and mass m must be substituted by their "running" or "effective" value $g_i(\mu)$ and $m(\mu)$ which are the solutions to the RGE equations (8,9).

As one approaches a fixed point of Eq. (8), the couplings on which the anomalous dimension γ_ϕ depends go to constant values g_i^*, and the anomalous dimension reaches a constant value; similarly, as a result of equation (6), the two-point correlation function $G_{Imp}(q; ...; g(\mu); \mu)$ goes (after Fourier transforming) into a function $f(u)$ of argument, $u = t/r^z$ (where z is the value at the fixed point of a different anomalous dimension, the so called dynamical exponent) times $r^{2\chi}$, where χ is related to γ_ϕ at the fixed point. Here, $r = |\vec{x} - \vec{z}|$, and $t = |t_x - t_z|$. In the neighborhood of any RG fixed point, it is easy to derive the scaling form of the Green functions of the field theory in terms of dynamic critical exponents χ, z as follows. Under independent rescaling of coordinates and the time $\vec{x} = s\vec{x}'$, $t = s^z t'$ (note: this kind of independent space and time scaling is needed when treating

non-relativistic theories such as arise for example in diffusion and growth processes) the field scales as $\phi(\vec{x}, t) = s^{\chi}\phi(\vec{x}', t')$, so it follows that

$$
\begin{aligned}
G(\vec{x}, t) &= \langle \phi(\vec{x}, t)\phi(\vec{0}, 0) \rangle \\
&= s^{2\chi} \langle \phi(s^{-1}\vec{x}, s^{-z}t)\phi(0, 0) \rangle, \\
&= s^{2\chi} G(s^{-1}\vec{x}, s^{-z}t) \\
&= r^{2\chi} f\Big(\frac{t}{r^z}\Big),
\end{aligned} \tag{16}
$$

where the last line follows from choosing $s \sim |\vec{x}|$, $r = |\vec{x}|$. The asymptotic behavior of $f(u)$ is given by [18]

$$
\lim_{u \to 0} f(u) \to \text{const.} \,, \quad \text{and} \quad \lim_{u \to \infty} f(u) \to u^{2\chi/z} \,. \tag{17}
$$

To aid our understanding of the relation between the coarse-graining scale R and the sliding (momentum) scale μ, we can use the above general solution (15) plus simple dimensional analysis. Let d_{φ} denote the canonical dimension of the field: $[\varphi] = \mu^{d_{\varphi}}$, then $p(\varphi)$ must have dimension $[p] = [\varphi^{-1}] = \mu^{-d_{\varphi}}$ expressed in units of the sliding scale μ. The mass dimension $[m] = \mu$, and assuming dimensionless couplings, then $[g_j] = \mu^0 = 1$. Then from (15) it is easy to prove that

$$
p_{Imp}(\varphi(\mu_0), \{g_i(\mu_0)\}, m(\mu_0), \mu_0; R) = \mu^{-d_{\varphi}} \, e^{-\frac{1}{2}\int_{\mu_0}^{\mu} \gamma_{\phi}(u)\, d\ln u} F\Big(\frac{\varphi(\mu)}{\mu^{d_{\varphi}}}, \frac{m(\mu)}{\mu}, g_j(\mu); \mu R\Big), \tag{18}
$$

where F is a *dimensionless* function of the *dimensionless* arguments as wriiten here. Cast in this form, we can investigate the infrared limit of the probability density by taking $\mu \to 0$. The connection to the Wilsonian RG approach and the window scale is the following. In the Wilsonian RG, the degrees of freedom are coarse-grained either in real space or in momentum space, the latter typically proving to be the more technically convenient *choice*. This is performed over a finite-width momentum shell corresponding to $\Lambda/s \le |\vec{k}| \le \Lambda$; this corresponds in fact to a "top-hat" window in momentum space. The UV cutoff Λ is contracted down to Λ/s, where $s > 1$. The infrared limit obtains by taking $s \to \infty$. The connection with the field theory sliding scale is $\mu = \Lambda/s$ [19]. Therefore, when coarse-graining in real space, the contracted cutoff corresponds to an increasing length scale: $R = s/\Lambda$. So, the IR limit $\mu \to 0$ corresponds to $R \to \infty$. The dimensionless product $\mu R = 1$, and we can now replace $\mu = 1/R$ everywhere in (18). In the neighborhood of an IR fixed

point, reached by taking $R \to \infty$, we therefore obtain the asymptotic scaling form:

$$\lim_{R \to \infty} p_{Imp}(\varphi(\mu_0), \{g_i(\mu_0)\}, \ m(\mu_0) \ , \mu_0; R) = R^{d_\varphi + \frac{1}{2}\gamma_\phi(g^*)}$$

$$\times \quad F\Big(\frac{\varphi(\mu_0)}{\mu_0^{d_\varphi}}\big(\frac{\mu}{\mu_0}\big)^{-d_\varphi - \frac{1}{2}\gamma_\phi(g^*)}, \frac{m(\mu_0)}{\mu_0}\big(\frac{\mu}{\mu_0}\big)^{\delta(g^*)-1}, g_j^*\Big), \quad (19)$$

where $\mu_0 = 1/R_0$ is some reference scale and we have used the solutions of (8,9) at the fixed point g_j^* in arriving at this final form. At the fixed point, the exponential prefactor in (18) scales as $(\mu/\mu_0)^{-\frac{1}{2}\gamma_\phi(g^*)}$.

Note that (19) demonstrates *in general* that the large scale asymptotic form of the coarse-grained probability density goes as a non-trivial power of the window size R times a certain dimensionless function. The asymptotic behavior is controlled by the canonical dimension d_φ of the field, its anomalous dimension at the IR fixed point $\gamma_\phi(g^*)$ (as well as by $\delta(g^*)$ in a massive theory). Specification of p_{Imp} at some reference scale $R_0 = 1/\mu_0$ yields the explicit mathematical form of F. The derivation of the RG equation for p_{Imp} in (13), and the large-distance scaling behavior of its general solution in (19) are the key results of this Letter.

To illustrate the use of the above *general* results, we apply them to two simple examples: (i) a free particle in the presence of a heat bath, and (ii) a system described by the Kardar-Parisi-Zhang (KPZ) equation with colored noise [20].

In the case of a free particle in a heat bath there is no self-interaction, but the statistics of the bath turns the problem into a classical Brownian motion problem; for the KPZ system there are interactions among the particles making up the system (described by the KPZ field) and also interactions with the bath (which could be an external environment or the effective result of a "microscopic" dynamics) represented by a noise term that drives the time derivative of the KPZ field. For each of these examples we will obtain the form of equation (18) which corresponds to the scale–dependent form of the probability $p_{Imp}(\varphi; R)$. To lowest order, the probability density is gaussian, thus the procedure consists in computing the improved form of $G(\vec{x}, \vec{z}; t_x, t_z)$ appearing in Eq. (5) and given by Eq. (14). We indicate briefly how to RG-improve non-gaussian probabilities below.

Example (i). The equation of motion for a point particle (a particle may be regarded as a zero-dimensional field) in a medium with viscous drag and subject to a random force is

$$\dot{\vec{v}}(t) = -\gamma \vec{v}(t) + \eta(t). \quad (20)$$

Here P of Eq. (4) is [14]

$$P[r] \propto \exp \left\{ -\frac{1}{4} \int_0^t ds \left(\frac{d\vec{r}}{ds} \right)^2 \right\} ,$$ (21)

where $\vec{r}(t)$ denotes the coordinate of the particle at time t and $\vec{v}(t)$ is the particle's velocity. The object $G(t, t')$ is

$$G_{ij}(t, t') = \langle r_i(t) r_j(t') \rangle \propto t \, \delta_{ij} \delta(t - t') .$$ (22)

There are no corrections due to fluctuations or interactions, and the probability of Eq. (5) is simply given by inserting Eq. (22) into (5) (after using a temporal "top-hat" window W_T)

$$p(r) \propto \frac{1}{T^{d/2}} \cdot \exp \left(-\frac{r^2}{2T} \right) ,$$ (23)

(where d is the number of components of the vector $\vec{r}$) which of course is the probability distribution that a particle executing standard Brownian motion be at position $r = |\vec{r}|$ at time T.

Example (ii). The KPZ equation (24) is a non-linear Langevin equation for a field. Contrast this to (20) which is a linear Langevin equation for a point-particle. In a system described by the KPZ equation

$$+ \frac{\partial \phi}{\partial t} = \nu \nabla^2 \phi + \frac{1}{2} \lambda (\nabla \phi)^2 + \eta(x, t)$$ (24)

with colored noise [21] $\eta(\vec{x}, t)$, there are corrections due to both fluctuations and interactions. Setting $\lambda = 0$ yields the linear Edwards-Wilkinson (EW) model having a unique IR fixed point $P1$, for which the IR critical exponents are known exactly for all space dimensions: $(z, \chi) = (2, (2 - d)/2)$, in the case of white uncorrelated noise [22]. The EW model is the free-field limit of the KPZ equation and generalizes the concept of random walk of a classical particle to the level of free fields. For non-zero λ two new fixed points arise and their corresponding critical exponents χ and z have values differing from the EW fixed point exponents. Briefly, in $d = 3$ space dimensions $P1$ is a saddle point with exponents $(z, \chi) = (2, -\frac{1}{2})$, $P2$ is infrared unstable with $(z, \chi) = (\frac{13}{6}, -\frac{1}{6})$, and $P3$ is infrared stable with $(z, \chi) = (\frac{2}{3}, \frac{4}{3})$. For each of these fixed points one has to consider two possibilities (cf.

9

Eq. (17) above), depending on whether $|t - t'| \gg |\vec{r} - \vec{r}'|^z$ or $|t - t'| \ll |\vec{r} - \vec{r}'|^z$ since, as mentioned above, these limits lead to different asymptotic behaviors for the scaling function f in (17). The corrected form of the PDF that the effective field has *value* $\bar{\varphi}$ is then obtained by inserting the "improved" form of G, Eq. (16), into Eq.(5); coarse-graining it with the window function W_R and using (15) leads to

$$
p_{Imp}(\bar{\varphi}; R)
$$
$$
= e^{-\frac{1}{2} \int_{\mu_0}^{\mu} \gamma_\phi(u) \, d\ln u} \sqrt{\frac{V^2}{2\pi [W_R \cdot G_{Imp} \cdot W_R]}} \cdot \exp\left(-\frac{\bar{\varphi}^2}{\frac{2}{V^2}[W_R \cdot G_{Imp} \cdot W_R]}\right).
$$

To proceed further we need the scaling form of the coarse-grained improved two-point function. To this end, we take a window function of the simple form $W_{R,T}(x) = \Theta(R - |\vec{x}|)\Theta(T - t_x)$ and without loss of generality, take the window center at the origin. (The *scaling* cannot and does not depend on where the window is located.) Then we find that

$$
\frac{1}{V^2}[W_{R,T} \cdot G_{Imp} \cdot W_{R,T}] \sim R^{2\chi}, \tag{25}
$$

for $|t| \ll |r|^z$ and

$$
\frac{1}{V^2}[W_{R,T} \cdot G_{Imp} \cdot W_{R,T}] \sim T^{2\chi/z}, \tag{26}
$$

for $|t| \gg |r|^z$, respectively.

For each of the fixed points the asymptotic limits are

$$
\lim_{|t| \gg |r|^z} p_{Imp}(\bar{\varphi}; R, T) \sim T^{-\chi/z + \frac{1}{2}\gamma_\phi(g^*)/z} \exp\left(-\frac{1}{2}\frac{\bar{\varphi}^2}{T^{2\chi/z}}\right) \tag{27}
$$

and

$$
\lim_{|t| \ll |r|^z} p_{Imp}(\bar{\varphi}; R, T) \sim R^{-\chi + \frac{1}{2}\gamma_\phi(g^*)} \exp\left(-\frac{1}{2}\frac{\bar{\varphi}^2}{R^{2\chi}}\right). \tag{28}
$$

The roughness exponent and the anomalous dimension are related through the exponent identity $\chi = -d_\varphi - \frac{1}{2}\gamma_\phi(g^*)$ which follows from using (14) plus dimensional analysis to arrive at a scaling form for G_{Imp} in complete analogy to what we worked out above for p_{Imp} in (18) and in (19). Comparing the scaling form so obtained with (16) immediately yields this identity.

We see that in the KPZ problem the asymptotic probability distributions associated with the fixed points $P2$, and $P3$ are the ones for *fractal* Brownian motion [23], unlike in the

10

free *field* case, $\lambda = 0$, or EW model, which describes Brownian motion for non-interacting fields.

In comparing KPZ with EW results, we have fractal Brownian motion in time, Eq (27), and in space Eq. (28), because the exponent combinations $2\chi/z$ and 2χ appearing within the exponential function are different from the EW values at the fixed points of the RGE P2 and P3.

Moreover, a nonzero wavefunction renormalization $\gamma_\phi(g^*) \neq 0$ can modify the exponents of the power-law prefactors in (27) and (28) away from their naive canonical values. For EW, the anomalous dimension is identically zero. The reason for the appearance of fractal Brownian motion in Example (ii) is now obvious: the combined effect of fluctuations and interactions drives χ and z away from their free field theory (EW) values and the probability distribution is correspondingly shifted from regular Brownian motion to fractal Brownian motion.

RG-improvement and the asymptotic scaling of non-gaussian probabilities can be worked out using the techniques of this paper. For any interacting field theory, one can expand the exact characteristic functional about the gaussian limit and proceed to derive the associated RG equation for p_{Imp} by substituting the identity

$$Z[J] = \exp\left(S_{interaction}[\frac{\delta}{\delta J}]\right) Z_{Gaussian}[J]|_{J \to \frac{i\xi}{V} W_R}, \tag{29}$$

directly into the definition of p_{Imp} in (3). Here, $S_{interaction}$ represents the non-Gaussian part of the action. In our examples, we have calculated the lowest order correction, which is the RG improved Gaussian. Non-Gaussian terms will appear at higher order in the couplings and these will take the form of a polynomial in φ times a gaussian. The polynomial will depend on higher n-point Green functions (e.g., for $n \geq 2$), each of which obeys Eq(6).

In summary we see that the *combined* action in space-time evolution of interactions and fluctuations leads to probability distributions which at some scale may be gaussian. However, given a particular dynamics, as the spatial extent of the system or the time window over which the system is observed, is changed, the two-point correlation function acquires an anomalous dimension and the effective PDF becomes the one for fractional Brownian Motion (fBM). This fBM is commonly associated with complex behavior, such as known to occur in the financial markets, ecology or in river systems [1], and its character depends on the initial values assigned to the relevant couplings at some scale. We have developed

the general framework and carried out the above analysis for two complementary problems. For the free field, as expected, we recover Brownian motion; this is useful as a very simple test of the validity of the application of our framework; in the much more complex case of KPZ we find a wealth of behaviors, with *persistence* (related to *superdiffusive* processes) *antipersistence* (related to *subdiffusive* processes) and regular Brownian Motion [24]. In fact, our analysis explicitly shows that the nature of the fBM displayed by the field system is related to the choice of initial conditions for the couplings, since the values of the roughness and dynamical exponents χ and z depend on which basin of attraction of the fixed points the initial couplings are chosen. Thus, the notion of complexity is scale dependent as well as dependent on the initial conditions; furthermore, since a given dynamics may have a variety of fixed points, we see explicitly that a particular system may display various complex behaviors which reflect the presence of different environments.

Finally, we note that the methods presented here can be extended and applied to any system which can be described by a field theory, such as fluids, materials, aggregates, and so forth.

[1] See, e.g., special issue of Science on "Complex Systems", *Science* **284**, 2 April 1999.

[2] H. Scher, M. Shlesinger and J. T. Bendler, *Physics Today* **44** (1991) 26.

[3] N. Goldenfeld and L.P. Kadanoff, *Science* **284** (1999) 87.

[4] S.A. Kauffman, *The Origins of Order: Self-Organization and Selection in Evolution*, Oxford University Press, New York, 1993.

[5] B.B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco, 1982.

[6] M. Gell-Mann, *The Random and the Regular*, Santa Fe Institute's 1999 Ulam's Lecture Series, Santa Fe, NM, November 1999.

[7] S. Weinberg, *The Quantum Theory of Fields I & II*, Cambridge University Press, Cambridge,

England, 1996.

[8] K. Wilson and M.E. Fisher, *Phys. Rev. Lett.* **28** (1972) 248.

[9] M. Gell-Mann and F.E. Low, *Phys. Rev.* **95** (1954) 1300.

[10] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Addison-Wesley, New York, 1992.

[11] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York 1979.

[12] R. J. Rivers, *Path Integral Methods in Quantum Field Theory*, Cambridge University Press, Cambridge 1987.

[13] D. Hochberg, C. Molina-París, J. Pérez-Mercader and M. Visser, *Phys. Rev.* E**60** (1999) 6343.

[14] J. Zinn–Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press, Oxford, 1996, 3rd edition.

[15] D. Amit, *Field Theory, the Renormalization Group and Critical Phenomena*, World Scientific, Singapore, 1984, 2nd edition.

[16] L.S. Brown, *Quantum Field Theory*, Cambridge University Press, Cambridge, 1992.

[17] The term "improved" means [9], [7] and [10], that the effects of interactions and fluctuations have been taken into account. This is so on account of the fact that the coefficients in equations (6) and (13) are already calculated at each order of perturbation theory. The objects p_{Imp} and $G^{(2)}_{Imp}$ contain much more information.

[18] E. Frey and U. C. Tauber, *Phys. Rev.* **E50** (1994) 1024.

[19] M. Le Bellac, *Quantum and Statistical Field Theory*, Oxford University Press, Oxford, 1991; Chap 7.

[20] E. Medina, T. Hwa, M. Kardar and Y.-C. Zhang, *Phys. Rev. A* **39** (1989) 3053.

[21] The noise is defined to be gaussian, with zero mean and a two–point correlation function given by $\langle \eta(\vec{x}, t) \eta(\vec{x}', t') \rangle = 2D(\vec{x}, \vec{x}'; t, t')$. The local function $D(\vec{x}, \vec{x}'; t, t')$ is the double Fourier transform of $D(\vec{k}, \omega) \propto k^{-2\rho} \omega^{-2\theta}$. The existence of power law behavior for the correlation functions of ϕ is *not* due to the choice of noise correlations given above; it is a *generic* prediction of the Renormalization Group due to the fact that the RGEs are quasi–linear first order partial differential equations.

[22] A.-L. Barabási and H.E. Stanley, *Fractal Concepts in Surface Growth*, Cambridge University Press, Cambridge, 1995.

[23] K. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1986.

[24] M. Shlesinger, G. Zaslavsky and U. Frisch, *Lévy Flights and Related Topics in Physics*, Lecture Notes in Physics, 1995, Springer–Verlag.